

# A solvable model for excitonic complexes in one dimension

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## Abstract

It is known experimentally that stable few-body clusters containing negatively-charged electrons (e) and positively-charged holes (h) can exist in low-dimensional semiconductor nanostructures. In addition to the familiar exciton (e+h), three-body ‘charged excitons’ (2e+h and 2h+e) have also been observed. Much less is known about the properties of such charged excitons since three-body problems are generally very difficult to solve, even numerically. Here we introduce a simple model, which can be considered as an extended Calogero model, to calculate analytically the energy spectra for both a charged exciton and a neutral exciton in a one-dimensional nanostructure, such as a finite-length quantum wire. Apart from its physical motivation, the model is of mathematical interest in that it can be related to the Heun (or Heine) equation and, as shown explicitly, highly accurate, closed form solutions can be obtained.

## I. Introduction

The optical properties of low-dimensional semiconductor structures, called nanostructures, have attracted much attention in the past few years, both experimentally and theoretically. One of the most interesting questions concerns the properties of ‘excitons’ in such low-dimensional structures. An exciton ( $X$ ) is a neutral, two-body complex formed by the attractive force between a negatively-charged electron ( $e$ ) in the semiconductor conduction band and a positively-charged hole ( $h$ ) in the valence band. An exciton therefore appears to be somewhat analogous to a hydrogen atom. There are, however, two important distinctions. First, the hole and electron masses are typically of the same order of magnitude and, second, the low dimensionality of the nanostructure can restrict the electron and hole motion to such an extent that the exciton must be treated as either two- or one-dimensional.

Recent observations of anomalies in the optical spectra of quantum wells (i.e. two-dimensional nanostructures) have been attributed to the formation of negatively-charged excitons<sup>1</sup>. Such complexes can arise when an exciton is created in the presence of a low concentration of free electrons; it may then be energetically favorable for the exciton to capture one electron to form a negatively-charged exciton, i.e.  $X + e \rightarrow X^-$ . In addition to two-dimensional nanostructures, it is interesting to consider the possibility of  $X^-$  formation in one-dimensional nanostructures such as a quantum wire. The consideration of such low-dimensional systems is particularly important since the exciton binding energies in a quantum wire are higher than those in the quantum well, owing to the reduced dimensionality; this increased exciton binding energy is thought to underlie the recently observed exciton lasing in a quantum wire device<sup>2</sup>.

This paper uses a simple model to investigate the properties of three-body complexes such as charged excitons. In particular, we provide closed-form expressions for the energy spectra of both a charged exciton  $X^-$  and a neutral exciton  $X$  in a finite-length, one-dimensional quantum wire. Even in one dimension, the three-body problem (i.e. two electrons ( $e$ ) and one hole ( $h$ )) with a Coulomb interaction would necessitate a computationally-intensive numerical solution. This is not our goal; instead we wish to demonstrate that analytically-

solvable models can be introduced to identify trends in the  $X^-$  and  $X$  energy spectra as a function of device parameters. We therefore sacrifice quantitative accuracy concerning a particular device in favour of a broader understanding of qualitative behaviour.

Our model considers an inverse-square interaction potential between particles (e-h is attractive and e-e is repulsive) together with a parabolic confinement potential of arbitrary strength along the wire. A non-Coulombic form for the interaction is in fact not unrealistic in nanostructures due to the presence of image charges in neighbouring gates and electrodes (see, for example, Ref. 3). As will be discussed, the energy spectrum of the neutral exciton  $X$  shows the same qualitative behaviour with both  $\frac{1}{x^2}$  (i.e. inverse-square) and  $\frac{1}{|x|}$  (i.e. Coulomb) interactions. For  $X^-$ , the three-body Schrodinger equation is shown to reduce to the Heun equation. The complete energy spectrum is found in the regime of physical interest. The analysis suggests that the  $X^-$  complex can have an enhanced stability as compared to  $X$ .

## II. Neutral exciton $X$

Our model Hamiltonian for the neutral exciton  $X$  (i.e. e-h pair) in a finite-length one-dimensional quantum wire is given by

$$H = -\frac{\hbar^2}{2m^*} \left( \frac{\partial^2}{\partial x_e^2} + \frac{\partial^2}{\partial x_h^2} \right) + \frac{1}{2}m^*\omega_0^2(x_e^2 + x_h^2) - \frac{q_{eh}\hbar^2}{2m^*} \frac{1}{(x_e - x_h)^2}, \quad (1)$$

where  $x_e$  and  $x_h$  are the electron and hole coordinates, the dimensionless parameter  $q_{eh}$  characterizes the electron-hole interaction strength, and  $m^*$  is the effective mass of the electrons and holes (assumed identical). The parabolic confinement potential has arbitrary strength and is assumed to be the same for both the electron and hole; the confinement parameter  $\omega_0$  can be chosen so as to mimic the effect of a wire of finite length  $L$  since  $L^2 \sim \hbar(m^*\omega_0)^{-1}$ . If  $q_{eh} > \frac{1}{2}$  in Eq. (1) then it is trivial to show that the two particles collapse toward each other for any finite value of  $E$ ; this can be seen by examining the behavior of the wave function near  $x_e \simeq x_h$ . To avoid this we assume that  $0 < q_{eh} < \frac{1}{2}$ . In terms of the center-of-mass coordinate  $X = \frac{1}{2}(x_e + x_h)$  and the relative coordinate  $x = x_e - x_h$ , the

Hamiltonian becomes  $H = H_{c.m.}(X) + H_{rel}(x)$ , where  $H_{c.m.}$  is the Hamiltonian for a single particle in a one-dimensional harmonic potential and

$$H_{rel} = -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + \frac{1}{2}\mu\omega_0^2 x^2 - \frac{q_{eh}\hbar^2}{4\mu} \frac{1}{x^2} \quad (2)$$

with a reduced mass  $\mu = \frac{m^*}{2}$ . The eigenvalues of  $H_{c.m.}$  are the one-dimensional harmonic oscillator levels. The eigenvalues of  $H_{rel}$  are<sup>4</sup>

$$E_{rel}(n; q_{eh}) = \hbar\omega_0(2n + \frac{3}{2} - \Delta) \quad ; \quad n = 0, 1, 2, \dots \quad (3)$$

where

$$\Delta = \frac{1}{2} - \frac{1}{2}(1 - 2q_{eh})^{\frac{1}{2}}. \quad (4)$$

As the e-h interaction is reduced (i.e.  $q_{eh} \rightarrow 0$ ) Eq. (3) becomes

$$E(n; 0) = \hbar\omega_0(2n + \frac{3}{2}) \quad ; \quad n = 0, 1, 2, \dots \quad (5)$$

i.e. the odd energy levels of a harmonic oscillator<sup>4</sup>. We shall refer to the quantity  $\Delta$  as the electron-hole energy-shift for reasons which are clear by comparing Eq. (3) and Eq. (5).

The Hamiltonian for the neutral exciton with a Coulomb interaction,  $X_C$ , is the same as in Eq. (1), except that the last term is replaced by the interaction potential  $\frac{1}{|x|}$ . We emphasise, however, that in the presence of image charges in neighbouring gates etc.<sup>3</sup>, a Coulomb interaction will not necessarily be more realistic than an inverse-square interaction. To date, the one-dimensional Schrodinger equation for  $X_C$  has not been completely solved analytically. The Hamiltonian for  $X_C$  has the form

$$H_{rel}^C = -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + \frac{1}{2}\mu\omega_0^2 x^2 - \frac{e^2}{\epsilon_{eh}} \frac{1}{x} \quad (6)$$

where  $\epsilon_{eh}$  is the dielectric constant of the system.

Introducing the dimensionless coordinate  $y$ , where  $x = by$  with  $b = \sqrt{\frac{\hbar}{\mu\omega_0}}$ , the eigenvalue problem of Eq. (2) reduces to:  $-\psi''(y) + (y^2 - \frac{q_{eh}}{2} \frac{1}{y^2})\psi(y) = 2E_{rel}/\hbar\omega_0\psi(y)$ . Equation (6) reduces to:  $-\psi''(y) + (y^2 - \frac{2e^2}{\epsilon_{eh}} \sqrt{\frac{\mu}{\hbar^3\omega_0}} \frac{1}{y})\psi(y) = 2E_{rel}^C/\hbar\omega_0\psi(y)$ . Typical values for the device

parameters are as follows:  $\epsilon_{eh} = 12$ ,  $\hbar\omega_0 = 0.01eV$  and  $m^*/m = 0.07$ . Using these values, we have performed numerical calculations which show that the low-lying energy spectra for  $X_C$  and  $X$  can indeed be quantitatively similar, provided an appropriate value of the free parameter  $q_{eh}$  is chosen. In the  $q_{eh} \rightarrow 0$  limit  $X_C$  yields an *identical* energy spectrum to that given in Eq. (5). This indicates that the inverse-square interaction is 'as realistic' as the bare Coulomb interaction in the context of one-dimensional models. To understand why this should be so, one can consider trying to solve the  $X$  and  $X_C$  problems using the complete basis set of the one-dimensional oscillator eigenstates  $|n\rangle$  in the interval  $-\infty < x < \infty$ , i.e.

$$\langle x|n\rangle = A_n H_n(y) \exp(-y^2/2)$$

where  $y = \sqrt{m\omega_0/\hbar}x$ ;  $H_n$  is a Hermite polynomial and  $A_n$  is the appropriate normalisation constant. In the complete basis of kets  $|n\rangle$  we need to evaluate matrix elements of the type  $\langle m|H|n\rangle$ , where  $H$  is either the Hamiltonian of  $X$  or  $X_C$ . This matrix element includes terms like

$$A_n A_m \int_{-\infty}^{\infty} H_n(y) H_m(y) V(y) \exp(-y^2) dy$$

where the potential  $V(y)$  is either proportional to  $\frac{1}{x^2}$  or  $\frac{1}{|x|}$ . It is clear that the integral diverges when  $m$  and  $n$  are both even. As discussed in Ref. 6, this integral is only finite if we truncate our Hilbert space so that  $m$  and  $n$  are odd; this finding is hence consistent with the result of only odd energy levels in Eq. (5). In this sense, both the  $\frac{1}{|x|}$  and  $\frac{1}{x^2}$  potentials are 'non-penetrable'<sup>7</sup>. Physically, this means that the electrons and hole in the  $X$  and  $X_C$  complexes cannot interchange their particle ordering along the wire; the exchange energy is therefore zero. The configuration with  $x > 0$  is totally separate from the configuration with  $x < 0$ . Each energy level of  $X$  and  $X_C$  is therefore doubly degenerate when the complexes are defined in the full interval  $-\infty < x < \infty$ . Equation (3) represents the complete energy spectrum of the neutral exciton  $X$  with inverse-square interaction.

### III. Charged exciton $X^-$

Our proposed Hamiltonian for the  $X^-$  complex may be considered as a generalisation of the Hamiltonian discussed by Calogero<sup>4</sup>. Calogero considered the three-body problem with a harmonic oscillator potential and inverse-square pair potentials for the case of three identical particles. Our model Hamiltonian  $H$  for two electrons and one hole has the same form, but we allow the strength and sign of the interaction between the particles to be different. In particular,

$$H = \sum_{i=e1,e2,h} \left( -\frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} m^* \omega_0^2 x_i^2 \right) - \frac{q_{eh}\hbar^2}{2m^*} \frac{1}{(x_{e1} - x_h)^2} + \frac{q_{ee}\hbar^2}{2m^*} \frac{1}{(x_{e1} - x_{e2})^2} - \frac{q_{eh}\hbar^2}{2m^*} \frac{1}{(x_h - x_{e2})^2} \quad (7)$$

where  $x_{e1}$ ,  $x_{e2}$  and  $x_h$  are the coordinates of the two electrons and hole,  $q_{ee}$  is the electron-electron interaction parameter ( $q_{ee} > 0$ ); as in Sec. II we restrict the electron-hole interaction parameter to  $0 < q_{eh} < \frac{1}{2}$ . The three-body problem in Eq. (7) is separable<sup>4</sup>. The separation involves two coordinate transformations; first a Jacobi transformation:  $X = \frac{1}{3}(x_{e1} + x_{e2} + x_h)$ ,  $x = 2^{-1/2}(x_{e1} - x_h)$  and  $y = 6^{-1/2}(x_{e1} + x_h - 2x_{e2})$  which enables us to rewrite  $H = H_{c.m.}(X) + H_{rel}(x, y)$ . Second, we write  $H_{rel}$  in terms of polar coordinates  $x = r \sin(\phi - \pi/3)$  and  $y = r \cos(\phi - \pi/3)$ :

$$H_{rel}(r, \phi) = -\frac{\hbar^2}{2m^*} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{1}{2} m^* \omega_0^2 r^2 + \frac{\hbar^2}{2m^*} \frac{M}{r^2}, \quad (8)$$

where  $M$  is the following operator

$$M = -\frac{\partial^2}{\partial \phi^2} + \frac{1}{2} \left[ -\frac{q_{eh}}{\sin^2(\phi - \pi/3)} + \frac{q_{ee}}{\sin^2(\phi)} - \frac{q_{eh}}{\sin^2(\phi + \pi/3)} \right]. \quad (9)$$

Writing an eigenfunction of Eq. (8) as  $\psi(r, \phi) = R(r)\Phi(\phi)$ , the three-body problem has therefore been reduced to finding the solutions of the following two ordinary, second-order differential equations

$$H_{rel}R(r) = E_{rel}R(r) \quad (10)$$

and

$$M\Phi_l(\phi) = b_l^2\Phi_l(\phi). \quad (11)$$

The eigenvalue problem in Eq. (10) is solved in Ref. 4 and the eigenvalues are given by

$$E_{rel} = \hbar\omega_0(2n + 1 + b_l) \quad ; n = 0, 1, 2, \dots \quad (12)$$

leaving the nontrivial problem of solving Eq. (11). For a given particle configuration the angle  $\phi$  is limited to a certain interval. The ordering of the three particles (i.e. 2e + 1h) is therefore determined by  $\phi$ . As seen earlier for X, the ‘non-penetrable’ interaction potentials prevent particle interchange and therefore make it necessary to treat different particle configurations separately. In the interval  $\phi \in ]0; \pi[$  the relationship between  $\phi$  and the particle configuration is given by

$$\begin{aligned} \phi \in ]0; \pi/3[ & : x_{e2} < x_{e1} < x_h \\ \phi \in ]\pi/3; 2\pi/3[ & : x_{e2} < x_h < x_{e1} \\ \phi \in ]2\pi/3; \pi[ & : x_h < x_{e2} < x_{e1}. \end{aligned} \quad (13)$$

Using the trigonometric identity  $\sin 3\phi = 3\sin\phi - 4\sin^3\phi$  we can rewrite the wave function in Eq. (11) as

$$\Phi(x) = (x - 1)^{\Delta_{ee}/2}(x - \frac{1}{4})^{\Delta_{eh}}y(x) \quad (14)$$

where  $x = \cos^2(\phi)$ ,  $\Delta_{eh} = 1 - \Delta = 1/2 + 1/2(1 - 2q_{eh})^{1/2}$  and  $\Delta_{ee} = 1/2 + 1/2(1 + 2q_{ee})^{1/2}$ . Hence we have reduced the problem of solving Eq. (11) to solving the Heun differential equation<sup>8</sup>

$$\frac{d^2y}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-1/4} \right) \frac{dy}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-1/4)}y = 0, \quad (15)$$

with coefficients

$$\begin{aligned} \gamma &= 1/2 & \alpha &= \Delta_{ee}/2 + \Delta_{eh} - 1/2 b_l \\ \delta &= \Delta_{ee} + 1/2 & \beta &= \Delta_{ee}/2 + \Delta_{eh} + 1/2 b_l \\ \epsilon &= 2\Delta_{eh} & q &= 1/2(\Delta_{eh})^2 + (\Delta_{ee}/4)^2 - (b_l/4)^2 \end{aligned} \quad (16)$$

and with the five parameters satisfying

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0 . \quad (17)$$

$q$  is the so-called accessory parameter. An introduction to the general features of Heun's equation is given in Ref.<sup>9</sup>. In addition, a recent bibliography containing 300 classified entries related to Heun's equation are listed in Ref.<sup>10</sup>. The relation in Eq. (17) ensures that the four singularities of Eq. (15) stay regular. One of these regular singularities is, however, an elementary singularity (since  $\gamma = 1/2$ ) which implies that Eq. (15) is a special case of the Heun equation called the Heine equation<sup>11</sup>. To our understanding, Heine's equation is far less well-known and we will therefore continue referring to Eq. (15) as the Heun equation. Solutions to Eq. (15) which are of particular interest are those which are analytic in some domain enclosing two singularities. Such solutions are called 'Heun functions' and are denoted by  $H_f$ , using the notation of Ref.<sup>9</sup>. We employ this notation in order to distinguish Heun functions  $H_f$  from Heun polynomials  $H_p$ , which are analytic in an interval containing *three* singularities. Below, Heun functions which are analytic in the interval  $x \in [0; 1/4]$  or  $\phi \in [\pi/3; 2\pi/3]$  will be studied in more detail. An eigenfunction of Eq.(15) defined in the region  $\phi \in [\pi/3; 2\pi/3]$  can be written, using  $\cos^2\phi = 1 - \sin^2\phi$ , as

$$\Phi_l(\phi) = (\sin\phi)^{\Delta_{ee}} (\sin^2\phi - 3/4)^{\Delta_{eh}} H_f(1/4, q; \alpha, \beta, \gamma, \delta; \cos^2\phi) . \quad (18)$$

Heun functions may be found by the power-series method or by the method of hypergeometric function series<sup>9</sup>. However for both methods the coefficients in the series have to satisfy a three-term recursion relation. It might be possible to use the method of augmented convergence to extract Heun functions from such relations, but in most cases only a numerical procedure is possible<sup>12</sup>. Although we cannot in general solve the eigenvalue problem of Eq. (15) analytically, we will now obtain a set of highly accurate, approximate solutions for the region  $\phi \in [\pi/3; 2\pi/3]$ ; these solutions are essentially exact for a large interval of the ratio  $q_{ee}/q_{eh}$ , including the range of physical interest. Employing the following trigonometric formula

$$-\frac{f}{\sin^2(\phi - \pi/3)} + \frac{g}{\sin^2(\phi)} - \frac{f}{\sin^2(\phi + \pi/3)} = \frac{f + g}{\sin^2(\phi)} - \frac{9f}{\sin^2(3\phi)} , \quad (19)$$

where  $f$  and  $g$  are arbitrary functions<sup>13</sup>, the operator  $M$  in Eq. (11) reduces to

$$M = -\frac{\partial^2}{\partial\phi^2} + \frac{q_{eh}}{2} \left[ \frac{1+\kappa}{\sin^2\phi} - \frac{9}{\sin^2 3\phi} \right] \quad (20)$$

where  $\kappa = q_{ee}/q_{eh}$ . Consider the angular dependence of the total interaction potential of the  $X^-$  complex:  $V(\phi) \equiv (1+\kappa)/\sin^2\phi - 9/\sin^2 3\phi$ . Here  $V(\phi)$  is periodic in  $\phi$ ,  $V(\phi) = V(\phi+\pi)$ , hence the physics of the three-body problem is contained within a  $\phi$ -interval of  $\pi$ , e.g.  $\phi \in ]0; \pi[$ . Figure 1 shows  $V(\phi)$  for the case  $\kappa = 1$ . The asymptotic behavior of  $V(\phi)$  is easily understood: at  $\phi = 0$  the repulsion between the two electrons causes a positive singularity, at  $\phi = \pi/3$  the attraction between the electron at  $x_{e1}$  and the hole at  $x_h$  causes a negative singularity, etc. For the particle configuration where the hole is between the two electrons (in Fig. 1:  $\phi \in ]\pi/3; 2\pi/3[$ ), the potential does not have a repulsive divergence and therefore corresponds to the most stable three-body configuration. A very good approximation to  $V(\phi)$  in this interval ( $\phi \in ]\pi/3; 2\pi/3[$ ) for moderate values of  $\kappa$  is

$$V_{app}(\phi) = 1 + \kappa - \frac{9}{\sin^2 3\phi} \quad (21)$$

This is illustrated in Fig. 2. Evidently the approximation becomes more exact for smaller  $\kappa$  and, as will become clearer later, exact in the limit  $q_{ee} \rightarrow -q_{eh}$ . We shall restrict  $\kappa$  to the interval  $\kappa \in [0; 20[$ ;  $V_{app}(\phi)$  is now an excellent approximation to  $V(\phi)$ . The quantity  $\kappa$  is the ratio of the strength of the electron-electron interaction to the strength of the electron-hole interaction, and for all practical electronic devices we expect this ratio to be less than 20. Using this approximation, Eq. (11) becomes  $M_{app}\Phi_l(\phi) = b_l^2\Phi_l(\phi)$  where  $M_{app} = -\partial^2/\partial\phi^2 + \frac{q_{eh}}{2}V_{app}$ . The task of solving this differential equation can now be transformed into the problem of solving a hypergeometric equation. The solutions are

$$\Phi_l(\phi) = (\sin 3\phi)^{\Delta_{eh}} {}_2F_1(a, b; c; \cos^2 3\phi) \quad (22)$$

where

$$c = 1/2, \quad a = 1/2(\Delta_{eh} - b_l'), \quad b = 1/2(\Delta_{eh} + b_l') \quad (23)$$

and  $b_l'^2 = 1/9 b_l^2 - 1/18(q_{eh} + q_{ee})$ . The exact eigenvalues are found to be

$$b_l = [9(l+1-\Delta)^2 + \frac{1}{2}(q_{ee} + q_{eh})]^{1/2} \quad ; \quad l = 0, 1, \dots \quad (24)$$

Combining Eq. (24) and Eq. (12) gives the complete energy spectrum of the  $X^-$  complex for the particle configuration where the hole is placed between the two electrons and  $\kappa \in [0; 20[$ ; in particular,

$$E_{rel}(n, l; q_{eh}, q_{ee}) = \hbar\omega_0 \left( 2n + 1 + [9(l+1-\Delta)^2 + \frac{1}{2}(q_{ee} + q_{eh})]^{\frac{1}{2}} \right) . \quad (25)$$

In the Calogero-model limit where  $q_{ee} \rightarrow -q_{eh}$ , the energy spectrum in Eq. (25) reduces to the energy spectrum found by Calogero for three identical particles<sup>4</sup>. By comparing Eq. (22) and Eq. (18) the approximation corresponding to Eq. (21) can be expressed as

$$\begin{aligned} \Phi_l(\phi) &= (\sin\phi)^{\Delta_{ee}} (\sin^2\phi - 3/4)^{\Delta_{eh}} Hf(1/4, q; \alpha, \beta, \gamma, \delta; \cos^2\phi) \\ &\simeq (\sin 3\phi)^{\Delta_{eh}} {}_2F_1(a, b; c; \cos^2 3\phi) \end{aligned} \quad (26)$$

In the Calogero-model limit, Eq.(26) becomes exact and  $Hf = (-4)^{\Delta_{eh}} {}_2F_1$ .

There exist two distinct particle configurations in which the hole is positioned between the two electrons. These two configurations represent different physically-accessible systems, because of the ‘non-penetrable’ property of the interaction potential as discussed in Sec. II. If  $X^-$  is defined for the full range of these two particle configurations, then each level in Eq. (25) is doubly-degenerate. By forming suitable linear combinations, the subspace corresponding to each eigenvalue may be spanned by an antisymmetric and symmetric wave function with respect to interchange of the two electrons.

#### IV. Comparison of energies of $X$ and $X^-$

We have derived the energy spectra for  $X$  and  $X^-$ . One possibility might be to compare their relative stabilities by turning off the confinement potential and calculating the binding energies of  $X$  and  $X^-$ . However, a finite confinement potential is needed within the present model in order to produce discrete energy levels for the complexes; when the confinement interaction is turned off ( $\omega_0 \rightarrow 0$ ) a continuous spectrum<sup>14</sup> is obtained for both  $X$  and  $X^-$ .

This suggests that  $X$  and  $X^-$  are not exciton complexes in the usual sense, since their existence depends on the presence of a confinement potential. Keeping the confinement potential finite, the ground state energies of  $X$  and  $X^-$  in the non-interacting limit ( $q_{ee}, q_{eh} \rightarrow 0$ ) are given by  $(E_{rel} + E_{c.m.})$  which yields  $2\hbar\omega_0$  and  $9/2\hbar\omega_0$  respectively; these energies are identical to the ground state energies for two and three spinless fermions in a harmonic potential well. We emphasize that this particular non-interacting limit is reached as a consequence of the model being strictly one-dimensional and containing a singular potential.

Keeping the confinement potential finite, we now investigate the *changes* in energy of  $X$  and  $X^-$  as the two-body interaction is varied. We introduce a quantity which we refer to as the ‘interaction energy’  $E_{int}$ , defined as the energy obtained by subtracting the total energy with vanishingly small interactions (i.e.  $q_{ee}, q_{eh} \rightarrow 0$ ) from the total energy with finite interactions. For  $X$  and  $X^-$  in their respective ground-states, we obtain

$$E_{int}^X(q_{eh}) = E_{rel}(0; q_{eh}) - E_{rel}(0; 0) = -\hbar\omega_0\Delta \quad (27)$$

and

$$E_{int}^{X^-}(q_{ee}, q_{eh}) = E_{rel}(0, 0; q_{ee}, q_{eh}) - E_{rel}(0, 0; 0, 0) = \hbar\omega_0 \left( [9(1 - \Delta)^2 + \frac{1}{2}(q_{ee} + q_{eh})]^{1/2} - 3 \right) . \quad (28)$$

As expected, both  $E_{int}^X$  and  $E_{int}^{X^-}$  become increasingly negative with increasing  $q_{eh}$ , i.e. the ground-state energy decreases as the electron-hole interaction  $q_{eh}$  increases. In addition  $E_{int}^{X^-}$  increases as the electron-electron interaction  $q_{ee}$  increases. We therefore interpret the quantities  $E_{int}^X$  and  $E_{int}^{X^-}$  as indicative of the binding strength of  $X$  and  $X^-$  respectively. As a consequence, the relative stability of  $X^-$  and  $X$  is then effectively represented by the ‘relative binding strength’

$$\Delta E_{int}(q_{ee}, q_{eh}) = E_{int}^{X^-} - E_{int}^X = \hbar\omega_0 \left( [9(1 - \Delta)^2 + \frac{1}{2}(q_{ee} + q_{eh})]^{1/2} - 3 + \Delta \right) . \quad (29)$$

If  $\Delta E_{int} < 0$ , this would suggest that  $X^-$  is more strongly bound than  $X$ ; if  $\Delta E_{int} > 0$ , the reverse is true. The cross-over occurs when  $\Delta E_{int} = 0$ , i.e. when

$$q_{ee} = 8\Delta + 7q_{eh} . \quad (30)$$

We label the  $\kappa$  value for which Eq. (30) is satisfied to be  $\kappa_{eq}$ . When  $\kappa < \kappa_{eq}$  for a given  $q_{eh}$ , then  $\Delta E_{int} < 0$ ; when  $\kappa > \kappa_{eq}$  the reverse is true. This is illustrated in the inset of Fig. 3 where  $\Delta E$  is plotted as a function of  $\kappa$  for the case  $q_{eh} = 0.25$ . The main part of Fig. 3 shows  $\kappa_{eq}$  as a function of  $q_{eh}$ . Two features are interesting. First,  $\kappa_{eq}$  stays within a reasonably narrow interval  $11 < \kappa_{eq} < 15$  for all values of  $q_{eh}$  within the model. In the range  $0 < q_{eh} < 0.4$ ,  $\kappa_{eq}$  increases only slightly on an absolute scale. Second, the minimum value for  $\kappa_{eq}$  is given by  $\kappa_{eq} = 11$ . Therefore if  $\kappa$  is smaller than 11, as is the case for typical experimental devices, this would suggest that  $X^-$  is more strongly bound than  $X$ .

## V. Conclusions

In summary, we studied a new model for a class of three-body problems; highly accurate, closed form solutions for this model were obtained. The model was used to study the exciton complex  $X^-$  relative to the neutral exciton  $X$ . The analysis suggest that  $X^-$  might be more strongly bound than  $X$  for typical one-dimensional devices.

Finally we note that the present results for  $X^-$  apply equally well to  $X^+$  (i.e. one electron plus two holes). Future work will use the same model to examine more exotic excitonic complexes containing  $n_e$  electrons and  $n_h$  holes, i.e.  $X^{\Delta n} -$  where  $\Delta n = n_e - n_h$ .

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## REFERENCES

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<sup>12</sup> Software for PC compatible computers running Window 3.1 is available from Dr. S. Slavyanov. It gives a graphical representation of Heun's differential equations, together with some additional tools for handling them. See last page of S. Slavyanov, W. Lay and A. Seeger in *Heun's differential equation*.

<sup>13</sup> For the special case  $f = -g$ , the formula in Eq. (12) reduces to the trigonometric identity found by Calogero (Ref. [4]).

<sup>14</sup> For X, the solution to Eq. (2) with  $\omega_0 = 0$  is given by  $\psi(x) = x^{1/2} J_{1/2-\Delta}(kx)$  and  $E_{rel} = \frac{\hbar^2}{2\mu} k^2$  ( $0 \leq k < \infty$ ). Here  $J_{1/2-\Delta}$  is a Bessel function<sup>4</sup>. Parameter  $\Delta$  is given in Eq. (4).

## Figure Captions

Figure 1.

The potential function  $V(\phi)$  for the case  $\kappa = 1$ . Also shown schematically are the corresponding electron-hole configurations corresponding to the various  $\phi$ -regions.

Figure 2.

Comparison of the approximate potential  $V_{app}(\phi)$  and the exact potential  $V(\phi)$  for various values of  $\kappa$ . For the case  $\kappa = 1$ ,  $V(\phi)$  and  $V_{app}(\phi)$  are essentially identical.

Figure 3.

The ratio of the electron-electron interaction to the electron-hole interaction at the stability cross-over point,  $\kappa_{eq}$ , as a function of the electron-hole interaction  $q_{eh}$ . The inset shows  $\Delta E_{int}$  as a function of  $\kappa$  for the case  $q_{eh} = 0.25$ .





